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# Characterization of Perfect Ulam Permutation Codes via Spheres and Young Tableaux

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**Abstract**

Permutation codes in the Ulam metric have recently been proposed for use in flash memory devices. We explore the possibility of perfect permutation codes in the Ulam metric by considering sphere sizes of permutations in the Ulam metric. We also introduce new a method to calculate exact sphere sizes using Young Tableaux. The discussion is then extended to multipermutation codes, where we consider sphere sizes toward understanding perfect Ulam multipermutation codes.

## 1 Introduction

A permutation code $C$ is a subset of the symmetric group $\mathbb{S}_n$, equipped with a distance metric. The application of permutation codes and multipermutation codes for use in non-volatile memory storage systems such as flash memory has received attention in the coding theory literature in recent years [1, 8, 9, 10]. One of the main distance metrics in the literature has been the Kendall-$\tau$ metric, which is suitable for correction of the type of error expected to occur in flash memory devices. Errors occur in these devices when electric cell charges leak over time or there is an overshoot of charge level in the rewriting process. For relatively small leak or overshoot errors the Kendtall-$\tau$ metric is appropriate, but may not be well-suited for large errors within a single cell.

In 2013, Farnoud et al. proposed permutation codes using the Ulam metric [3]. They showed that the use of the Ulam metric would allow a large leakage or overshoot error within a single cell to be viewed as a single error. Subsequent papers expounded on the use of Ulam metric in multipermutation codes and bounds on the size of permutation codes in the Ulam metric [4, 7]. Meanwhile, Buzaglo et al. discovered the existence of a perfect permutation code under the cyclic Kendall-$\tau$ metric, and proved the non-existence of perfect permutation codes under the Kendall-$\tau$ metric for certain parameters [2].

In this paper we considerer four main questions. The first question is: How can Ulam sphere sizes be calculated? One answer to this question is to use Young Tableaux and the RSK-Correspondence (Lemma 3.2). The second question is: Do perfect Ulam permutation codes exists? The answer to this question is that nontrivial perfect Ulam permutation codes do not exist (Theorem 4.1). Both questions are closely related since perfect Ulam permutation code sizes are characterized by Ulam sphere sizes.

The discussion is then extended to multipermutation codes, where we consider the third question: How are are the Ulam metrics related for permutations and multipermutations, and how can the multipermutation Ulam metric be simplified? Lemmas 7.1 and 7.2 address this question. Finally, the fourth question is: How can multipermutation Ulam sphere sizes be calculated? A partial answer to this question is provided in Remark 8.1 and Lemma 8.5, which provide calculation methods for certain parameters.

The organization is as follows: Section 2 defines notation and basic concepts used in the paper. Section 3 introduces a method of calculating Ulam sphere size Young tableaux and the RSK-Correspondence. Section 4 focuses on proving the non-existence of perfect codes, and Section 5 briefly discusses a partial ordering of the symmetric group based on the Ulam metric.

The remaining sections focus on multipermutations. In Section 6, basic notations for multipermutations and the $r$-regular Ulam metric are introduced. In Section 7, the Ulam metric for permutations and the $r$-regular Ulam metric for multipermutaions are compared, and the $r$-regular Ulam metric is simplified. Finally, in Section 8, $r$-regular Ulam sphere sizes are discussed and in Section 9 concluding remarks are given.

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## 2 Preliminaries and Notation

In this paper we utilize the following notation and definitions, generally following conventions established in [3]. The symbol $[n]$ denotes the set of integers $\{1, 2, \ldots, n\}$. The symbol $\mathbb{S}_n$ stands for the set of permutations (automorphisms) on $[n]$, i.e., the symmetric group of order $n!$. For a permutation $\sigma \in \mathbb{S}_n$, we use the notation $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$, where for all $i \in [n]$, $\sigma(i)$ is the image of $i$ under $\sigma$. Under this notation, we may also view $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ as the sequence of $\sigma$, which we reference later in one of the equivalent definitions of the Ulam distance. Given two permutations $\sigma, \pi \in \mathbb{S}_n$, the product $\sigma\pi$ is defined by $(\sigma\pi)(i) = \sigma(\pi(i))$. In other words, we define multiplication of permutations by composition, e.g., $[2, 1, 5, 4, 3][5, 1, 4, 2, 3] = [3, 2, 4, 1, 5]$. The identity permutation $[1, 2, \ldots, n] \in \mathbb{S}_n$ is denoted by $e$.

**Definition** ($\phi(i, j)$, translocation). Given distinct $i, j \in [n]$, the symbol $\phi(i, j) \in \mathbb{S}_n$ is defined as follows:

$$\phi(i,j) := \begin{cases} [1, 2, \ldots i-1, i+1, i+2, \ldots, j, i, j+1, \ldots, n] & \text{if } i < j \\ [1, 2, \ldots j-1, i, j, j+1, \ldots, i-1, i+1, \ldots, n] & \text{if } i > j \end{cases}$$

If $i = j$, then define $\phi(i, j) := e$. We refer to $\phi(i, j)$ as a **translocation**, and if we do not specify the indexes $i$ and $j$ we may denote a translocation simply by $\phi$.

Intuitively, a translocation is the permutation that results in a delete/insertion operation. More specifically, given $\sigma \in \mathbb{S}_n$ and the translocation $\phi(i, j) \in \mathbb{S}_n$, the product $\sigma\phi(i, j)$ is the result of first removing $\sigma(i)$ from $\sigma$, then shifting all positions between $i$ and $j$, including $j$, by one (left if $i < j$ and right if $i > j$), and finally reinserting $\sigma(i)$ into the new $j$th position. Figure 1 illustrates the permutation $\sigma = [2, 4, 1, 5, 3]$ represented physically by relative cell charge levels and the effect of multiplying $\sigma$ by the translocation $\phi(1, 5)$ (top half of figure) or $\phi(4, 2)$ (bottom half of figure) on the right. Notice that multiplying by $\phi(1, 5)$ corresponds to the error that occurs when the highest (1st) ranked cell suffers charge leakage that results in it being the lowest (5th) ranked cell. Multiplying by $\phi(4, 2)$ corresponds to the error that occurs when the 4th highest cell is overfilled so that it is the 2nd highest cell.

We next define the Ulam distance. For the definition, it is first necessary to define $\ell(\sigma, \pi)$. Given permutations $\sigma, \pi \in \mathbb{S}_n$, then $\ell(\sigma, \pi)$ denotes the length of the longest common subsequence of $\sigma$ and $\pi$. More precisely, $\ell(\sigma, \pi)$ is the largest integer $m \in \mathbb{Z}_{>0}$ such that there exists a sequence $(a_1, a_2, \ldots, a_m)$ where for all $r \in [m]$, we have $a_r = \sigma(i_r) = \pi(j_r)$ with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq n$. The length of the longest increasing subsequence of a permutation $\sigma \in \mathbb{S}_n$ is $\ell(\sigma, e)$, denoted simply by $\ell(\sigma)$.

**Definition** ($\mathrm{d}_\circ(\sigma, \pi)$, Ulam distance). Given $\sigma, \pi \in \mathbb{S}_n$, then

$$\mathrm{d}_\circ(\sigma, \pi) := n - \ell(\sigma, \pi).$$

We call $\mathrm{d}_\circ(\sigma, \pi)$ the **Ulam distance** between $\sigma$ and $\pi$.

Note that $\mathrm{d}_\circ(\sigma, e) = n - \ell(\sigma)$. It is also well-known that $\mathrm{d}_\circ(\sigma, \pi)$ equals the minimum number of translocations needed to transform $\sigma$ into $\pi$ [3, 7]. That is, $\mathrm{d}_\circ(\sigma, \pi) = \min\{k \in \mathbb{Z}_{\geq 0} \mid \exists \phi_1, \phi_2, \ldots, \phi_k \text{ s.t. } \sigma\phi_1\phi_2 \cdots \phi_k = \pi\}$. We now define the notions of an $(n, M, d)$ code and an Ulam sphere.

**Definition** ($(n, M, d)$ code). A subset $C \subseteq \mathbb{S}_n$ is a called an $(n, M, d)$ **code** if and only if
1) $|C| = M$, and
2) $\min_{c_1, c_2 \in C, c_1 \neq c_2} \mathrm{d}_\circ(c_1, c_2) = d$.

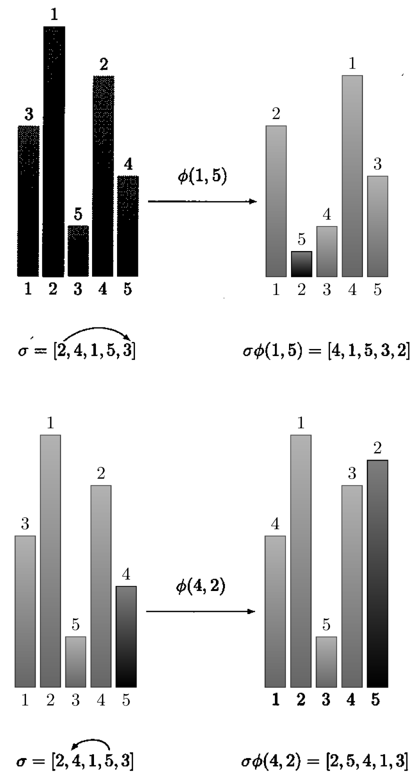
**Definition** ($S(\sigma, r)$, Ulam sphere). Given $\sigma \in \mathbb{S}_n$ and $t \in \{0, 1, \ldots, n-1\}$, we define

$$S(\sigma, t) := \{\pi \in \mathbb{S}_n \mid \mathrm{d}_\circ(c, \pi) \leq t\},$$

and call $S(\sigma, t)$ the **Ulam sphere centered at** $\sigma$ **of radius** $t$, or simply the **sphere centered at** $\sigma$ **of radius** $t$

In the definition we use $t$ for the radius instead of $r$ because $r$ is reserved for the repetition number in $r$-regular multipermutations discussed later. It is known that an $(n, M, d)$ code is $t$-error correcting if and only if $d \geq 2t + 1$ [3]. This is because if the distance between two codewords is greater or equal to $2t + 1$, then after $t$ or fewer errors (multiplication by $t$ or fewer translocations), the resulting permutation remains closer to the original permutation than any other permutation. Finally, we define perfect codes.

Figure 1: Translocation illustration




**Definition** (perfect code). a subset $C \subseteq \mathbb{S}_n$ is called a **perfect $t$-error correcting Ulam permutation code** (or simply a **perfect code** if the context is clear) if and only if for any $\sigma \in \mathbb{S}_n$ there exists a unique codeword $c \in C$ such that $\sigma \in S(c,t)$.

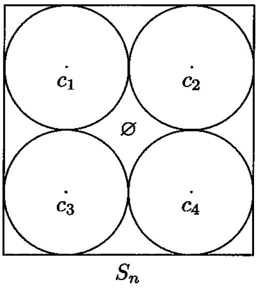
A perfect code partitions $\mathbb{S}_n$ as illustrated in Figure 2. In the figure four codewords $c_1, c_2, c_3$, and $c_4$ are shown with their spheres combining to fill $S_n$. The area outside of these spheres is empty so that any member of $S_n$ is within one of the spheres. A perfect code $C \subseteq \mathbb{S}_n$ is said to be **trivial** if either (1) $C = \mathbb{S}_n$ (occurring when $t = 0$); or (2) $|C| = 1$ (occurring when $t = n-1$).

# 3 Ulam Sphere Size

Perfect codes and sphere sizes are related as follows: a perfect $t$-error correcting code $C \subseteq S_n$, if it exists, will have cardinality $|C| = \frac{n!}{S(c,t)}$, where $c \in C$. Hence one of the first questions that may be considered in exploring the possibility of a perfect code is the feasibility of a code of such size. Toward that end, the calculation of the size of the sphere $S(c,t)$ will prove to be useful. As noted in [3], for any $\sigma \in \mathbb{S}_n$, we have $|S(\sigma,t)| = |S(e,t)|$. Hence calculation of Ulam sphere sizes can be reduced to the case when the identity is the center.

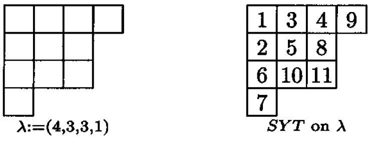
One way to calculate Ulam sphere sizes centered at $e$ is to use Young tableaux and the RSK-Correspondence. It is first necessary to introduce some basic notation and definitions regarding Young tableaux. Additional information on the subject can be found in [6] and [12].

Figure 2: Perfect code illustration




First, a *Young diagram* is a left-justified collection of cells with a (weakly) decreasing number of cells in each row below. Listing the number of cells in each row gives a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$, where $n$ is the total number of cells in the Young diagram. The notation $\lambda \vdash n$ is used to mean $\lambda$ is a partition of $n$. Because the partition $\lambda \vdash n$ defines a unique Young diagram and vice versa, a Young diagram may be referred to by its associated partition $\lambda \vdash n$. For example, the partition $\lambda := (4,3,3,1) \vdash 11$ has the corresponding Young diagram pictured on the left side of Figure 3.

Figure 3: Young diagram and SYT




A *standard Young tableau*, abbreviated by *SYT*, is a filling of a Young diagram $\lambda \vdash n$ with the following three qualities: (1) cell values are strictly increasing across each row; (2) cell values are strictly increasing down each column; and (3) each of the integers 1 through $n$ appears exactly once. One possible $SYT$ on $\lambda := (4,3,3,1)$ is pictured in the right side of Figure 3. Following conventional notation ([6, 12]), $f^\lambda$ denotes the number of $SYT$ on $\lambda \vdash n$. The next lemma, a stronger form of which appears in [6], is an application of the famous RSK-correspondence.

**Lemma 3.1.** *Let $\sigma \in S_n$ and let $P$ and $Q$, both on $\lambda \vdash n$, be the pair of SYT associated with $\sigma$ by the RSK-correspondence.*

$$\text{Then } \lambda_1 = \ell(\sigma).$$

The above lemma states that the number of columns of $P$ (equivalently $Q$) associated with $\sigma$ by the RSK-correspondence is equal to the length of the longest increasing subsequence of $\sigma$. Applying this lemma to calculate Ulam sphere sizes, we are able to relate Ulam sphere sizes to $f^\lambda$.

**Lemma 3.2.** *Let $t \in \{0, 1, \ldots, n-1\}$ and $\Lambda := \{\lambda \vdash n \mid \lambda_1 \geq n - t\}$.*

$$\text{Then } |S(e,t)| = \sum_{\lambda \in \Lambda} (f^\lambda)^2.$$

*Proof.* Assume that $t \in \{0, 1, \ldots, n-1\}$, and let $\Lambda := \{\lambda \vdash n \mid \lambda_1 \geq n - t\}$. By definition of an Ulam sphere, $|S(e,t)| = |\{\pi \in \mathbb{S}_n \mid \mathrm{d}_\circ(e,\pi) \leq t\}|$, which is equal to $|\{\pi \in \mathbb{S}_n \mid n - \ell(\pi) \leq t\}|$ by the definition of Ulam distance. This in turn is trivially equal to $|\{\pi \in \mathbb{S} \mid \ell(\pi) \geq n - t\}|$, which by Lemma 3.1 equals the number of ordered pairs $(P, Q)$ of $SYT$ on $\lambda \vdash n$ where $\lambda \in \Lambda$. In other words, $|\{\pi \in \mathbb{S} \mid \ell(\pi) \geq n - t\}| = |\{(P, Q) \text{ on } \lambda \vdash n \mid \lambda \in \Lambda\}|$. Finally, the conclusion holds by definition of $f^\lambda$. $\square$

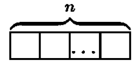
The formula below, known as the *hook length formula*, is due to Frame, Robinson, and Thrall [5, 6]. In the formula, the notation $(i,j) \in \lambda$ is used to refer to the cell in the $i$th row and $j$th column of a Young diagram $\lambda \vdash n$. The notation $h(i,j)$ denotes the *hook length* of $(i,j) \in \lambda$, i.e., the number of boxes below or to the right of $(i,j)$, including the box $(i,j)$ itself. More formally, $h(i,j) := |\{(i,j^*) \in \lambda \mid j^* \geq j\} \cup \{(i^*,j) \in \lambda \mid i^* \geq i\}|$. The formula is as follows:

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}.$$

Applying Lemma 3.2 and the hook length formula provides an answer to the first main question of calculating Ulam sphere sizes. Sizes can be explicitly calculated as demonstrated in the following lemmas. These lemmas will be useful later to show the nonexistence of nontrivial $t$-error correcting perfect codes for $t \in \{1,2,3\}$. In the calculations we implicitly use the fact mentioned earlier that sphere sizes can be reduced to the case when $e$ is the center.

**Lemma 3.3.** *Let $\sigma \in \mathbb{S}_n$. Then $|S(\sigma,0)| = 1$.*

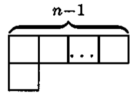
*Proof.* Although this is an obvious fact, we wish to consider why it is true from the perspective of Lemma 3.2. Note first that there is only one partition $\lambda \vdash n$ such that $\lambda_1 = n$, namely $\lambda := (n)$ with the associated Young diagram below.




It is clear that there is only one possible Young tableau on $\lambda$ so that $(f^\lambda) = 1$, and thus by Lemma 3.2 $|S(\sigma,0)| = 1$. □

**Lemma 3.4.** *Let $\sigma \in \mathbb{S}_n$. Then $|S(\sigma,1)| = 1 + (n-1)^2$.*

*Proof.* There is only one possible partition $\lambda \vdash n$ such that $\lambda_1 = n-1$, namely $\lambda := (n-1,1)$, with its Young diagram pictured below.



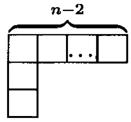

Therefore by Lemma 3.2 and the previous example, $|S(\sigma,1)| = 1 + (f^\lambda)^2$. Applying the hook length formula, we obtain $(f^\lambda)^2 = \left(\frac{n!}{(n)(n-2)!}\right)^2 = (n-1)^2$, which implies that $|S(\sigma,1)| = 1 + (n-1)^2$. □

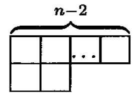
The same method used in the proof of Lemma 3.4 may be applied to spheres of larger radii. As alluded to in Theorem 3.2, the cardinality of the set $\{\pi \in \mathbb{S}_n \mid \ell(\pi) = n-t\}$ is exactly the sum of the squares of the number of standard tableaux on distinct Young diagrams with $n-r$ columns. Moreover, the size of a sphere of radius $t$ is equal to the size of a sphere of radius $t-1$ plus $|\{\pi \in \mathbb{S}_n \mid \ell(\pi) = n-t\}|$.

**Lemma 3.5.** *Let $\sigma \in \mathbb{S}_n$ and $n > 3$. Then*

$$|S(\sigma,2)| = 1 + (n-1)^2 + \left(\frac{(n)(n-3)}{2}\right)^2 + \left(\frac{(n-1)(n-2)}{2}\right)^2.$$

*Proof.* Note first that $|S(\sigma,2)| = |S(\sigma,1)| + |\{\pi \in \mathbb{S}_n \mid \ell(\pi) = n-2\}|$. The only partitions $\lambda \vdash n$ such that $\lambda_1 = n-2$ are $\lambda^{(1)} := (n-2,1,1)$ and $\lambda^{(2)} := (n-2,2)$, with their respective Young diagrams pictured below.



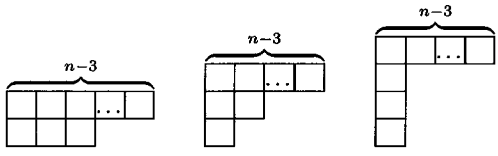

Using the hook length formula, $f^{\lambda^{(1)}}$ and $f^{\lambda^{(2)}}$ may be calculated to yield: $f^{\lambda^{(1)}} = \frac{(n)(n-3)}{2}$ and $f^{\lambda^{(2)}} = \frac{(n-1)(n-2)}{2}$. Following the same reasoning as in Lemma 3.4 yields the desired result. □

Such individual cases could be considered indefinitely. However, the following sphere is the last instance of significance in this paper as its result will be necessary to prove the main theorem, that nontrivial perfect codes in the Ulam metric do not exist.

**Lemma 3.6.** *Let $\sigma \in \mathbb{S}_n$ and $n > 5$. Then*

$$\begin{aligned}|S(\sigma,3)| &= 1+(n-1)^2+\left(\frac{(n)(n-3)}{2}\right)^2+\left(\frac{(n-1)(n-2)}{2}\right)^2\\ &+ \left(\frac{(n)(n-1)(n-5)}{6}\right)^2+\left(\frac{(n)(n-2)(n-4)}{3}\right)^2+\left(\frac{(n-1)(n-2)(n-3)}{6}\right)^2.\end{aligned}$$

*Proof.* The proof is essentially the same as the proofs for Lemmas 3.4 and 3.5. In this case $|\{\pi \in \mathbb{S}_n \mid \ell(\pi) = n-3\}|$ can be calculated by considering the partitions $\lambda^{(1)} := (n-3,3)$, $\lambda^{(2)} := (n-3,2,1)$, and $\lambda^{(3)} := (n-3,1,1,1)$, the only Young diagrams having $n-3$ columns. These Young diagrams are pictured below.




Applying the hook length formula to $\lambda^{(1)}$, $\lambda^{(2)}$, and $\lambda^{(3)}$ and adding the value from Lemma 3.5 yields the result.

□

# 4 Nonexistence of Nontrivial Perfect Ulam Codes

In 2013, Farnoud et. al ([3]) proved the following upper bound on the size of an $(n,M,d)$ code $C$:

$$M \leq (n-d+1)! \tag{1}$$

Hence one strategy to prove the non-existence of perfect codes is to show that the size of a perfect code must necessarily be larger than the upper-bound given above. Note that for equation (1) to make sense, $d$ must be less than or equal to $n-1$. This is always true since the maximum Ulam distance between any two permutations in $\mathbb{S}_n$ is $n-1$, achieved when permutations are in reverse order of each other (e.g., $\mathrm{d}_\circ(e,[n,n-1,...,1]) = n-1$).

**Lemma 4.1.** *There do not exist any (nontrivial) single-error correcting perfect codes.*

*Proof.* Assume that $C \subset \mathbb{S}_n$ is a perfect single-error correcting code. Recall that $C$ is trivial code if either $C = \mathbb{S}_n$ or if $|C| = 1$. If $n \leq 2$, then for all $\sigma, \pi \in \mathbb{S}_n$, we have $\pi \in S(\sigma,1)$, which implies that $C$ is a trivial code. Thus we may assume that $n > 2$.

We proceed by contradiction. Since $C$ is a perfect single-error correcting code, then $C$ is an $(n,M,d)$ code with $3 \leq d \leq n-1$ and $M = \frac{n!}{|S(\sigma,1)|} = \frac{n!}{1+(n-1)^2}$ by Lemma 3.4. However, inequality (1) implies that the code size $M \leq (n-2)!$. Hence, it suffices to show that $M = \frac{n!}{1+(n-1)^2} > (n-2)!$, which is true if and only if $n > 2$. □

Similar arguments may also be applied to show that no nontrivial perfect $t$-error correcting codes exist for $t \in \{2,3\}$. The remaining cases are treated toward the end of this section.

**Lemma 4.2.** *There do not exist any (nontrivial) perfect 2-error correcting codes.*

*Proof.* Assume that $C$ is a perfect 2-error correcting code. Similarly to the proof of Lemma 4.1, if $n \leq 3$, then $C$ is a trivial code consisting of a single element, so we may assume $n > 3$. Again we proceed by contradiction.

Since $C \subseteq \mathbb{S}_n$ is a perfect 2-error correcting code, then $C$ is an $(n,M,d)$ code with $5 \leq d \leq n-1$ and $M = \frac{n!}{|S(\sigma,2)|} = n!/\left[1+(n-1)^2+\left(\frac{(n)(n-3)}{2}\right)^2+\left(\frac{(n-1)(n-2)}{2}\right)^2\right]$ by Proposition 3.5. Inequality (1) implies that $M \leq (n-4)!$, so it suffices to prove that $f(n) := n!/\left[1+(n-1)^2+\left(\frac{(n)(n-3)}{2}\right)^2+\left(\frac{(n-1)(n-2)}{2}\right)^2\right] -$

$(n-4)! > 0$. Here $f'(n) = 2n^3 - 9n^2 + 9n - 1$, and $f''(n) = 6n^2 - 18n + 9$, which is positive for all $n > \frac{3+\sqrt{3}}{2} \approx 2.37$. Both $f'(4)$ and $f(4)$ are strictly greater than 0, which in turn implies that for all integer values of $n > 3$, it must be true that $f(n) > 0$. □

**Lemma 4.3.** *There do not exist any (nontrivial) perfect 3-error correcting codes.*

*Proof outline.* Assume that $C \subseteq \mathbb{S}_n$ is a perfect 3-error correcting code. Similarly to the proof of Lemmas 4.1 and 4.2, if $n \leq 7$, then $C$ is a trivial code, so we may assume that $n > 7$. The remainder of the proof follows the same reasoning as the proof for Lemma 4.2, utilizing the sphere size calculated in Lemma 3.6.

For small values of $t$, explicit sphere calculations work well for showing the non-existence of nontrivial perfect $t$-error correcting codes. However, for each radius $t$, the size of the sphere $S(e,t)$ is equal to $|S(e,t-1)| + |\{\pi \in \mathbb{S}_n \mid \ell(\pi) = n-t\}|$. This means each sphere size calculation of radius $t$ requires calculation of sphere sizes for radii from 0 through $t-1$. Hence such explicit calculations are impractical for large values of $t$. For values of $t > 6$, another method can be used to show that nontrivial perfect codes do not exist. The next lemma provides a sufficient condition to conclude that perfect codes do not exist. In the proof of the lemma, the notation $\binom{n}{t}$ denotes the usual combinatorial choice function, $\binom{n}{r}$ $:= \frac{n!}{(n-t)!t!}$.

**Lemma 4.4.** *Let $t, n \in \mathbb{Z}_{>0}$, and $t \leq n/2$. If the following inequality holds, then no perfect $t$-error correcting codes exist in $\mathbb{S}_n$ :*

$$F(n,t) := \frac{\left((n-t)!\right)^2 t!}{n!(n-2t)!} > 1. \tag{2}$$

*We call the above inequality the* ***overlapping condition.***

*Proof.* We proceed by contrapositive. Suppose $C \subset \mathbb{S}_n$ is a perfect $t$-error correcting code. Then $C$ is an $(n, M, d)$ code with $2t+1 \leq d \leq n-1$ and by inequality (1), $M \leq (n-2r)!$. At the same time, for any $\sigma \in \mathbb{S}_n$, we have $|S(\sigma,t)| = |S(e,t)|$, which is less than or equal to $\binom{n}{n-t}\frac{n!}{(n-t)!}$, since any permutation $\pi \in S(e,t)$ can be obtained by first choosing $n-r$ elements of $e$ to be in increasing order, and then arranging the remaining $t$ elements into $\pi$. Of course this method will generally result in double counting some permutations in $S(e,t)$, hence the inequality. Now $|S(\sigma,t)| \leq \binom{n}{n-t}\frac{n!}{(n-t)!}$ implies that $\frac{(n-t)!}{\binom{n}{t}} \leq \frac{n!}{|S(\sigma,t)|} = M \leq (n-2t)!$. Moreover, $\frac{(n-t)!}{\binom{n}{t}} \leq (n-2t)!$ if and only if $F(n,t) \leq 1$.

□

Notice that the overlapping condition is never satisfied for $t = 1$. However, the following proposition will imply that as long as $t > 1$, then the overlapping condition may be satisfied for sufficiently large $n$.

**Proposition 4.5.** *Let $t \in \mathbb{Z}_{>0}$ and $t \leq n/2$. Then $\lim_{n\to\infty} F(n,t) = t!$.*

*Proof.* Assume $t \in \mathbb{Z}_{>0}$ and $t \leq n/2$. Then

$$\begin{aligned}
\lim_{n\to\infty} F(n,t) &= \lim_{n\to\infty} \frac{(n-t)(n-t-1)\cdots(n-2t+1)(n-2t)!(n-t)!t!}{(n)(n-1)\cdots(n-t+1)(n-t)!(n-2t)!} \\
&= \lim_{n\to\infty} \frac{(n-t)(n-t-1)\cdots(n-2t+1)t!}{(n)(n-1)\cdots(n-t+1)} \\
&= \lim_{n\to\infty} \frac{(n^{t-1})t!}{n^{t-1}} = t!
\end{aligned}$$

□

The proposition above means that for any integer $t > 1$, there is some value $k$ such that for all values of $n$ larger than $k$, there does not exist a perfect $t$-error correcting code. The question remains of how large the value of $k$ must be before it is guaranteed that perfect $t$-error correcting codes do not exist.

Table 1 compares positive integer values $t$ versus $\min\{n \in \mathbb{Z}_{>0} \mid F(n,t) > 1\}$. Values were determined via numerical computer calculation. The table suggests that for $t > 6$, the minimum value of $n$ satisfying

Table 1: Non-feasibility of perfect $t$-error correcting codes

| $t$ | $\min n$ satisfying (2) | $t$ | $\min n$ satisfying (2) |
|---|---|---|---|
| 1 | N/A | 6 | 13 |
| 2 | 8 | 7 | 14 |
| 3 | 8 | 8 | 16 |
| 4 | 10 | 9 | 18 |
| 5 | 11 | 10 | 20 |

the overlapping condition is $n = 2t$. If what the table appears to suggest is true, then in view of Proposition 4.5, we may rule out perfect $t$-correcting codes for any $t > 6$. The next lemma formalizes what is implied in the table by providing parameters for which the overlapping condition is always satisfied. In combination with Lemma 4.4, the implication is that nontrivial perfect codes do not exist for these parameters. The remaining cases are also easily dealt with.

**Lemma 4.6.** *Let $t \in \mathbb{Z}$ and $t > 6$. Then $n \geq 2t$ implies that the overlapping condition is satisfied.*

*Proof.* Assume $t \in \mathbb{Z}$ and that $t > 6$. We begin the proof of the lemma by showing that if $n = 2t$, then the desired inequality holds. We assume that $n = 2t$ and proceed by induction on $t$.

For the base case, let $t = 7$. Then $n = 14$, and $F(n,t) = \frac{(7!)^3}{14!} \approx 1.46 > 1$. As the induction hypothesis, suppose it is true that $F(2t,t) = \frac{(t!)^3}{(tk)!} > 1$. We wish to show that the inequality $F(2(t+1), t+1) = \frac{((t+1)!)^3}{(2(t+1))!} > 1$ holds.

Here $\frac{((t+1)!)^3}{(2(t+1))!} = \left(\frac{(t!)^3}{(2t)!}\right)\left(\frac{(t+1)^3}{(2t+1)(2t+2)}\right)$. By our induction hypothesis, the first term of the right hand side is greater than 1, so it suffices to show that $\frac{(t+1)^3}{(2t+1)(2t+2)} \geq 1$. Note here that $\frac{(t+1)^3}{(2t+1)(2t+2)} > \frac{(t+1)^3}{(2t+2)(2t+2)} = \frac{(t+1)^3}{4(t+1)^2} = \frac{t+1}{4}$, which is greater than 1 whenever $t > 3$. Of course $t > 6$ by assumption, so the desired conclusion follows.

Thus far we have technically only proven that $F(n,t) > 1$ whenever $n = 2t$. However, it is a simple matter to show that the same is true whenever $n > 2t$ as well. We begin by supposing that $F(n,t) > 1$. Then $F(n+1,t) = \frac{((n+1-t)!)^2 t!}{(n+1)!(n+1-2t)!} = F(n,t) \cdot \frac{(n+1-t)^2}{(n+1)(n+1-2t)}$ is necessarily greater than 1 whenever $\frac{(n+1-t)^2}{(n+1)(n+1-2t)} \geq 1$, which is true for all values of $n$ and $t$. □

Lemma 4.6 required that $n \geq 2t$. However, if $n < 2t$, then it is impossible for a nontrivial perfect $t$-error correcting code to exist. In fact, we may say something even stronger.

**Remark 4.7.** If $n, t \in \mathbb{Z}_{>0}$ and $n \leq 2t+1$, then it is impossible for a nontrivial perfect $t$-error correcting code to exist.

To understand why Remark 4.7 is true, consider two permutations within $\mathbb{S}_n$ of maximal Ulam distance apart. The most obvious example of which would be the identity element $e$ and the only-decreasing permutation $\omega := [n, n-1, ..., 1]$. Notice that $S(e,t) = \{\pi \in \mathbb{S}_n \mid \ell(\pi) \geq n-t\}$, which means that every permutation whose longest increasing subsequence is at least $n-t$ is in the sphere centered at $e$. Meanwhile, there is at least one permutation $\sigma \in \mathbb{S}$ such that $\ell(\sigma) = 1+t$ and $\sigma \in S(\omega, t)$, since we may apply successive translocations to $\omega$ in such a way that the longest increasing subsequence is increased with each translocation. As long as $n \leq 2t+1$, then $n-t \leq t+1 = 1+t$, implying that $\ell(\sigma) = 1+t \geq n-t$, which implies that $\sigma \in S(e,t) \cap S(\omega,t)$. Therefore the only perfect code possible when $n \leq 2t+1$ is a single element code, i.e. a trivial code. Consolidating all previous results, we are now able to state the following generalized theorem. This is a main contribution of this paper.

**Theorem 4.1.** *There do not exist any nontrivial perfect codes in the Ulam metric.*

*Proof.* First, by Lemmas 4.1, 4.2, and 4.3, there do not exist any nontrivial perfect $t$-error correcting codes for $t \in \{1,2,3\}$. Next note that $F(n,r)$ increases as $n$ increases, and thus by numerical results (see Table 1), for all $t \in \{4,5,6\}$ the overlapping condition is satisfied whenever $n \geq 2t+2$. Therefore by Lemma 4.4, and Remark 4.7, there are no nontrivial perfect $t$-error correcting codes for $t \in \{4,5,6\}$. Finally, by Lemmas 4.4, 4.6, and Remark 4.7, there are no nontrivial perfect $r$-error correcting codes for $r > 6$. $\square$

# 5 Partial Ordering

The symmetric group, $\mathbb{S}_n$, may be viewed in terms of a partial ordering based on the Ulam metric. This partial ordering enables one to visualize $\mathbb{S}_n$ as a Hasse diagram where each row consists of permutations with a particular length of longest increasing subsequence. Such a diagram also gives a visual representation of the spheres centered at $e$, since each row corresponds to a set of permutations of a particular Ulam distance from $e$. Such visualization may help to choose a code as close to perfect as possible. The partial ordering is defined below.

**Definition** ($\leq_\circ$). Let $\sigma, \pi \in \mathbb{S}_n$. Then we say $\sigma <_\circ \phi$ if and only if the following two conditions are satisfied:
(1) There exists $m \in \mathbb{Z}_{>0}$ such that $\ell(\sigma) = \ell(\pi) + m$, and
(2) There exist translocations $\phi_1, \phi_2, \ldots, \phi_m$ such that $\pi = \sigma\phi_1\phi_2\cdots\phi_m$
and for $i = 1, \ldots, m$, we have $\ell(\sigma\phi_1\cdots\phi_{i-1}) = \ell(\sigma\phi_1\cdots\phi_i) + 1$.

A permutation $\sigma$ is $\leq_\circ$ another permutation $\pi$ if and only if $\sigma <_\circ \pi$ or $\sigma = \pi$.

Intuitively, the above definition simply means that $\sigma <_\circ \pi$ if $\pi$ can be obtained from $\sigma$ by applying a series of translocations, with the length of the longest increasing subsequence decreasing with each application of a translocation. The definition is defined in this way so that for all $\sigma \in \mathbb{S}_n$, if $\sigma \neq e$, then $e <_\circ \sigma$. It is easily verified that $\leq_\circ$ is indeed a partial ordering.

Reflexivity and antisymmetry both follow trivially from the definition. Transitivity is also a simple matter. If $a, b, c \in \mathbb{S}_n$ with $a \leq b \leq c$, and either $a = b$ or $b = c$, then transitivity follows immediately. Otherwise, if $a < b$ and $b < c$, then there exist integers $m$ and $k$ such that $\ell(a) = \ell(b) + m$ and $\ell(b) = \ell(c) + k$, with sets of translocations $\{\phi_1, \ldots, \phi_m\}$ and $\{\phi'_1, \ldots, \phi'_k\}$ such that $a\phi_1\cdots\phi_m = b$ and if $i \in [m]$, then $\ell(a\phi_1\cdots\phi_{i-1}) = \ell(a\phi_1\cdots\phi_i) + 1$, and similarly $b\phi'_1\cdots\phi'_k = c$ and if $i \in [k]$, then $\ell(b\phi'_1\cdots\phi'_{i-1}) = \ell(a\phi_1\cdots\phi_i) + 1$. Then $a <_\circ c$, since $\ell(a) = \ell(c) + (m+k)$ and the set of translocations $\{\phi_1, \ldots \phi_m, \phi'_1, \ldots, \phi'_k\}$ satisfies the second condition of the definition.

The partial ordering $\leq_\circ$ may be visualized in the form of a Hasse diagram. This allows, for example, the sphere $|S(e,r)|$ to be viewed simply as the bottom $r+1$ rows of the Hasse diagram for $\mathbb{S}_n$. Figure 4 shows the Hasse diagram for $\mathbb{S}_3$ under $\leq_\circ$ and part of the Hasse diagram for $\mathbb{S}_4$ under $\leq_\circ$. In the latter case, all connections to $[1,2,3,4]$, $[1,3,4,2]$, and $[4,3,2,1]$ in the partial ordering are depicted, but other connections are omitted for simplicity.

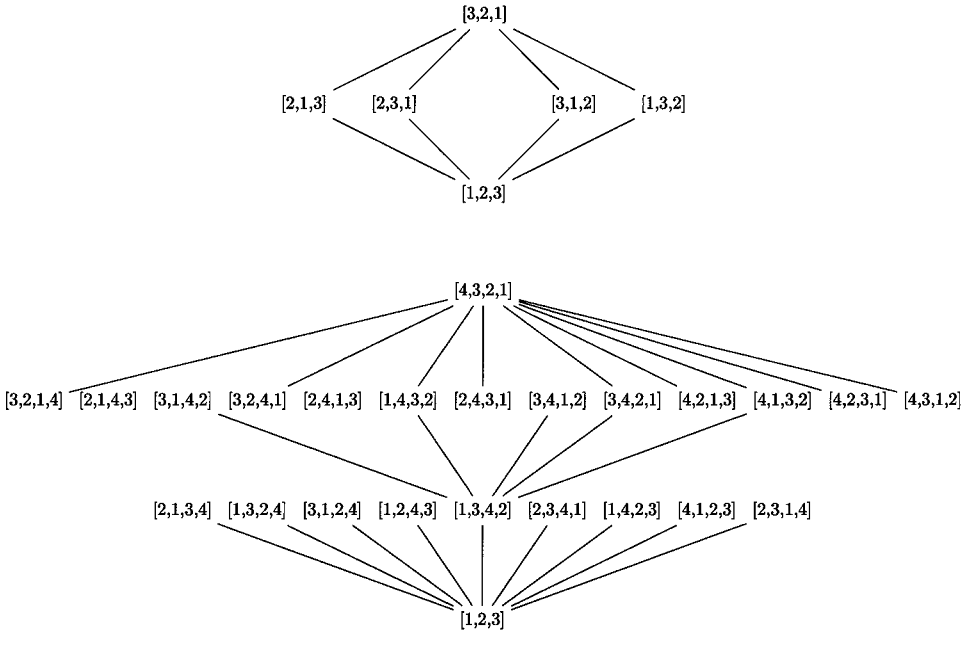
If $\sigma \leq_\circ \pi$, then we call $\pi$ a superior of $\sigma$. An interesting fact about the partial ordering is that there are permutations $\sigma \in \mathbb{S}$ for which there do not exist any superiors. This is of course trivially true for $\omega = [n, n-1, \ldots, 1]$, since this is always the permutation having the shortest longest increasing subsequence. However, there are also permutations having non-minimal longest increasing subsequences but without superiors. For example, the permutations $[2,1,4,3]$, $[3,1,4,2]$, $[2,4,1,3]$, and $[3,4,1,2]$ have no superiors, as can be clearly seen in the bottom half of Figure 4. This is generalized in the next proposition.

**Proposition 5.1.** *Let $k \in \mathbb{Z}_{>0}$ and $k \leq \frac{n}{2}$. Then there exist $\sigma \in \mathbb{S}_n$ such that $\ell(\sigma) = k$ and there do not exist any $\pi \in \mathbb{S}_n$ such that $\sigma <_\circ \pi$.*

*Proof.* Assume $k \in \mathbb{Z}_{>0}$ and $k \leq \frac{n}{2}$. Let $\sigma := [\underbrace{n, n-1, \ldots, 2k+1}_{n-2k}, \underbrace{2, 4, \ldots, 2k}_{k}, \underbrace{1, 3, \ldots, 2k-1}_{k}]$. Then $\ell(\sigma) = k$, but for all translocations $\phi$, we have $\ell(\sigma\phi) \geq k$, since either the even $k$-length increasing subsequence or the odd $k$-length increasing subsequence is retained in $\sigma\phi$. $\square$

The analogue to the proposition above for the case when $k > \frac{n}{2}$ is the following conjecture yet to be proven: If $k \in \mathbb{Z}$ and $k > \frac{n}{2}$, then for all $\sigma \in \mathbb{S}_n$ such that $\ell(\sigma) = k$, there exists $\pi \in \mathbb{S}_n$ such that $\sigma <_\circ \pi$. This would mean that there are always members in the upper half of the diagram without successors, but never such members in the lower half. The partial ordering and associated Hasse diagrams may provide a

Figure 4: Hasse diagram for $\mathbb{S}_3$ (top) and partial Hasse diagram for $\mathbb{S}_4$ (bottom)




useful tool for future analyzing of permutation codes in the Ulam metric, and perhaps assist in choosing code elements so that the chosen code is as large as possible.

# 6 Extension to Multipermutations

We now extend the discussion from permutation codes in the Ulam metric to multipermutation codes in the Ulam metric.. Specifically, we extend the discussion to $r$-regular multipermutations defined and studied in [4]. First, some basic notation and definitions are necessary. Unless otherwise stated, definitions are based on conventions established in [4].

An $r$-**regular multiset** is a multiset such that each of its element appears exactly $r$ times (i.e., each element is *repeated* $r$ times). Given positive integers $r$ and $n$ such that $r$ divides $n$, we write $\mathsf{M}(n,r)$ to denote the $r$-regular multiset whose elements are precisely the set $[n/r]$. For example, $\mathsf{M}(6,2) = \{1,1,2,2,3,3\}$. A **multipermutation** is a permutation of a multiset, and in the instance of an $r$-regular multiset, we call the multipermutation an $r$-**regular multipermutation**. For example, $(3,2,2,1,3,1)$ is a 2-regular multipermutation of $\mathsf{M}(6,2)$.

**Definition** ($\mathbf{m}_\sigma^r$). Let $n, r \in \mathbb{Z}_{>0}$ and $r|n$. Given an $r$-regular multiset $\mathsf{M}(n,r)$, for each $\sigma \in \mathbb{S}_n$ we define a corresponding $r$-regular multipermutation $\mathbf{m}_\sigma^r$ as follows:

for all $i \in [n]$ and $j \in [n/r]$,

$$\mathbf{m}_\sigma^r(i) := j \text{ if and only if } (j-1)r + 1 \le \sigma(i) \le jr,$$

and $\mathbf{m}_\sigma^r := (\mathbf{m}_\sigma^r(1), \mathbf{m}_\sigma^r(2), \ldots, \mathbf{m}_\sigma^r(n))$.

As an example of $\mathbf{m}_\sigma^r$, let $n = 6$, $r = 2$, and $\sigma = [1,5,2,4,3,6]$. Then $\mathbf{m}_\sigma^r = [1,3,1,2,2,3]$. Note that this definition differs slightly from the correspondence defined in [4], which was defined in terms of the inverse permutation. This is so that certain properties (discussed later) of the Ulam metric for permutations will also hold in the case of the Ulam metric for multipermutations. With the correspondence above, we may define an equivalence relation between elements of $\mathbb{S}_n$. For permutations $\sigma, \pi \in \mathbb{S}_n$, and $r$ dividing $n$, we say that $\sigma \equiv_r \pi$ if and only if $\mathbf{m}^r_\sigma = \mathbf{m}^r_\pi$. The equivalence class $R_r(\sigma)$ of $\sigma \in \mathbb{S}_n$ is defined by $R_r(\sigma) := \{\pi \in \mathbb{S}_n \mid \pi \equiv_r \sigma\}$. For a subset $S \subseteq \mathbb{S}_n$, the notation $\mathcal{M}_r(S) := \{\mathbf{m}^r_\sigma \mid \sigma \in S\}$, i.e. the set of $r$-regular multipermutations corresponding to elements of $S$.

The following definition is our own. We define the product $\mathbf{m}^r_\sigma \cdot \pi$ as $\mathbf{m}^r_\sigma \cdot \pi := \mathbf{m}^r_{\sigma\pi}$. Since it is possible for different permutations to correspond to the same multipermutation, we should clarify that $\mathbf{m}^r_\sigma = \mathbf{m}^r_\tau$ implies $\mathbf{m}^r_{\sigma\pi} = \mathbf{m}^r_{\tau\pi}$. Indeed this is true because if $\mathbf{m}^r_\sigma = \mathbf{m}^r_\tau$ then for all $i \in [n]$ we have $\mathbf{m}^r_\sigma(i) = \mathbf{m}^r_\tau(i)$, which implies for $j := \mathbf{m}^r_\sigma(i)$ that $(j-1)r + 1 \leq \sigma(i) \leq jr$ and $(j-1)r + 1 \leq \tau(i) \leq jr$. This in turn implies that $(j-1)r + 1 \leq \sigma\pi(\pi(i)) \leq jr$ and $(j-1)r + 1 \leq \tau\pi(\pi(i)) \leq jr$, which means $\mathbf{m}_{\sigma\pi}(\pi(i)) = \mathbf{m}_{\tau\pi}(\pi(i))$. Intuitively speaking, the same corresponding elements of the sequences $\sigma$ and $\tau$ still correspond (with a different index) after being multiplied on the right by $\pi$. Hence $\mathbf{m}^r_{\sigma\pi} = \mathbf{m}^r_{\tau\pi}$, or by our notation $\mathbf{m}^r_\sigma \cdot \pi = \mathbf{m}^r_\tau \cdot \pi$.

A quick example makes the explanation above clearer: if $n = 6$ and $r = 2$ with permutations $\sigma := [1, 2, 3, 4, 5, 6]$ and $\pi := [2, 1, 4, 3, 5, 6]$, then $\mathbf{m}^r_\sigma = \mathbf{m}^r_\pi = [1, 1, 2, 2, 3, 3]$. Here, if we take any permutation $\tau \in \mathbb{S}_6$ and multiply both $\sigma$ and $\pi$ on the right side by $\tau$, then in the resulting permutations $\sigma\pi$ and $\tau\pi$, we will still have the same corresponding values.

In flash memory devices, both permutations or multipermutations may be modeled physically by relative rankings of cell charges as depicted in Figure 1 of Section 2. The number of possible messages is limited by the number of distinguishable relative rankings. However, multipermutations may significantly increase the total possible messages compared to ordinary permutations. For example, if only $k$ different charge levels are utilized at a given time, then permutations of length $k$ can be stored. Hence, in $r$ blocks of length $k$, one may store $(k!)^r$ potential messages. On there other hand, if one uses $r$-regular multipermutations in the same set of blocks, then $(kr)!/(r!)^k$ potential messages are possible.

Recall that the Ulam metric $\mathrm{d}_\circ(\sigma, \pi)$ between permutations $\sigma, \pi \in \mathbb{S}_n$ was defined in terms of longest common subsequences: $\mathrm{d}_\circ(\sigma, \pi) := n - \ell(\sigma, \pi)$. Recall also that the Ulam distance $\mathrm{d}_\circ(\sigma, \pi)$ between $\sigma, \pi \in \mathbb{S}_n$ is equivalent to the minimum number of translocations needed to transform $\sigma$ into $\pi$. The $r$-regular Ulam distance for multipermutations is defined in terms of the Ulam distance for permutations.

**Definition** ($\mathrm{d}^r_\circ(\sigma, \pi)$, $r$-regular Ulam distance). Let $n, r \in \mathbb{Z}_{>0}$ and $r|n$. Also let $\sigma, \pi \in \mathbb{S}_n$. Define

$$\mathrm{d}^r_\circ(\sigma, \pi) := \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \mathrm{d}_\circ(\sigma', \pi').$$

We call $\mathrm{d}^r_\circ(\sigma, \pi)$ the **$r$-regular Ulam distance** between $\sigma$ and $\pi$.

Notice that the $r$-regular Ulam distance is defined over equivalence classes. That is, the definition is the minimum Ulam distance among all members of $R_r(\sigma)$ and $R_r(\pi)$. We will discuss this distance in more detail in the following section. We finish this section by defining what a multipermutation code is.

**Definition** ($r$-regular multipermutation code, MPC$(n, r)$). Given $r, n \in \mathbb{Z}_{>0}$ with $r|n$, then $C \subseteq \mathbb{S}_n$ is an **$r$-regular multipermutation code** if and only if for all $\sigma \in C$, we also have $R_r(\sigma) \subseteq C$. Such a code is denoted by MPC$(n, r)$, and we say that $C$ is an MPC$(n, r)$ code.

Because any time a permutation is a member of an MPC$(n, r)$ code $C$ its entire equivalence class is contained within $C$, then any MPC$(n, r)$ code $C$ can be represented by the set of $r$-regular multipermutations associated with elements of $C$, i.e. the set $\mathcal{M}_r(C)$. Moreover, the cardinality $|C|_r$ of an MPC$(n, r)$ code $C$ is defined as $|C|_r := |\mathcal{M}_r(C)|$ (this notation and definition differs slightly from [4]). Finally, an MPC$_\circ(n, r, d)$ **code** $C$ is an MPC$(n, r)$ code $C$ such that $\min_{\sigma, \pi \in C, \sigma \neq \pi} \mathrm{d}^r_\circ(\sigma, \pi) = d$.

# 7 Multipermutation Ulam Metric

In this section we discuss some similarities and differences between the Ulam metric for permutations and the Ulam metric for multipermutations. The $r$-regular Ulam distance, defined in the previous section, is a distance between equivalence classes. However, it is often convenient to think of the $r$-regular Ulam distance as a distance between multipermutations. Viewed this way, the property of the Ulam metric for permutations, that it can be defined in terms of longest common subsequences or equivalently in terms of translocations, carries over to the $r$-regular Ulam distance. The next lemma shows that the $r$-regular Ulam distance between permutations $\sigma$ and $\pi$ is equal to $n$ minus the length of the longest common subsequence of their corresponding $r$-regular multipermutations.

**Lemma 7.1.** *Let $r, n \in \mathbb{Z}_{>0}$, and $r|n$. Also let $\sigma, \pi \in \mathbb{S}_n$. Then*

$$\mathrm{d}_\circ^r(\sigma,\pi) = n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r).$$

*Proof.* Assume $r, n \in \mathbb{Z}_{>0}$, $r|n$, and $\sigma, \pi \in \mathbb{S}_n$. We will first show that $\mathrm{d}_\circ^r(\sigma,\pi) \geq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. By definition of $\mathrm{d}_\circ^r(\sigma,\pi)$, there exist $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ such that $\mathrm{d}_\circ^r(\sigma,\pi) = \mathrm{d}_\circ(\sigma',\pi') = n - \ell(\sigma',\pi')$. Hence if for all $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ we have $\ell(\sigma',\pi') \leq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$, then $\mathrm{d}_\circ^r(\sigma,\pi) \geq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$ (subtracting a larger value from $n$ results in a smaller overall value). Therefore it suffices to show that that for all $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$, that $\ell(\sigma',\pi') \leq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. This is simple to prove because if two permutations have a common subsequence, then their corresponding $r$-regular multipermutations will have related common subsequence. Let $\sigma' \in R_r(\sigma)$, $\pi' \in R_r(\pi)$, and $\ell(\sigma',\pi') = k$. Then there exist indexes $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ such that for all $p \in [k]$, $\sigma'(i_p) = \pi'(j_p)$. Of course, whenever $\sigma'(i) = \pi'(j)$, then $\mathbf{m}_{\sigma'}^r(i) = \mathbf{m}_{\pi'}^r(j)$. Therefore $\ell(\sigma',\pi') = k \leq \ell(\mathbf{m}_{\sigma'}^r, \mathbf{m}_{\pi'}^r) = \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$.

Next, we will show that $\mathrm{d}_\circ^r(\sigma,\pi) \leq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. Note that

$$\begin{aligned}\mathrm{d}_\circ^r(\sigma,\pi) &= \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \mathrm{d}_\circ(\sigma',\pi') \\ &= \min_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} (n - \ell(\sigma',\pi')) \\ &= n - \max_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \ell(\sigma',\pi').\end{aligned}$$

Here if $\max\limits_{\sigma' \in R_r(\sigma), \pi' \in R_r(\pi)} \ell(\sigma',\pi') \geq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$, then $\mathrm{d}_\circ^r(\sigma,\pi) \leq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$ (subtracting a smaller value from $n$ results in a larger overall value). It is enough to show that there exist $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ such that $\ell(\sigma',\pi') \geq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. To prove this fact, we take a longest common subsequence of $\mathbf{m}_\sigma^r$ and $\mathbf{m}_\pi^r$ and then carefully choose $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ to have an equally long common subsequence. The next paragraph describes how this can be done.

Let $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = k$ and let $(1 \leq i_1 < i_2 < \cdots < i_k \leq n)$ and $(1 \leq j_1 < j_2 < \cdots < j_k \leq n)$ be integer sequences such that for all $p \in [k]$, $\mathbf{m}_\sigma^r(i_p) = \mathbf{m}_\pi^r(j_p)$. The existence of such sequences is guaranteed by the definition of $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. Now for all $p \in [k]$, define $\sigma'(i_p)$ to be the smallest integer $l \in [n]$ such that $\mathbf{m}_\sigma(l) = \mathbf{m}_\sigma(i_p)$ and if $q \in [k]$ with $q < p$, then $\mathbf{m}_\sigma^r(i_q) = \mathbf{m}_\pi^r(i_p)$ implies $\sigma'(i_q) < \sigma'(i_p) = l$. For all $p \in [k]$, define $\pi(j_p)$ similarly. Then for all $p \in [k]$, $\sigma'(i_p) = \pi'(j_p)$. The remaining terms of $\sigma'$ and $\pi'$ may easily be chosen in such a manner that $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$. Thus there exist $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$ such that $\ell(\sigma',\pi') \geq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r)$. □

The following example helps to illuminate the choice of $\sigma'$ and $\pi'$ in the proof above. If $\mathbf{m}_\sigma^r = [2,1,2,1,3,3]$, and $\mathbf{m}_\pi^r = [3,2,2,1,3,1]$, then we have $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = 4$, with the common subsequence $(2,2,1,3)$ of maximal length. Here $(1,3,4,6)$ and $(2,3,4,5)$ are sequences with $\mathbf{m}_\sigma^r(1) = \mathbf{m}_\pi^r(2)$, $\mathbf{m}_\sigma^r(3) = \mathbf{m}_\pi^r(3)$, $\mathbf{m}_\sigma^r(4) = \mathbf{m}_\pi^r(4)$, and $\mathbf{m}_\sigma^r(6) = \mathbf{m}_\pi^r(5)$. Then following the convention outlined in the proof above, $\sigma'(1) = \pi'(2) = 3$, $\sigma'(3) = \pi'(3) = 4$, $\sigma'(4) = \pi'(4) = 1$, and $\sigma'(6) = \pi'(5) = 5$, so that $\ell(\sigma',\pi') \geq 4$. The other elements of $\sigma'$ and $\pi'$ can be chosen as follows so that $\sigma' \in R_r(\sigma)$ and $\pi' \in R_r(\pi)$: set $\sigma'(2) = 1$, $\sigma'(5) = 6$, $\pi'(1) = 1$, and $\pi'(6) = 6$.

If two multipermutations $\mathbf{m}_\sigma^r$ and $\mathbf{m}_\pi^r$ have a common subsequence of length $k$, then $\mathbf{m}_\sigma^r$ can be transformed into $\mathbf{m}_\pi^r$ with $n-k$ (but no fewer) delete/insert operations. Delete/insert operations correspond to applying (multiplying on the right) a translocation. Hence by the previous lemma we can state the following lemma about the $r$-regular Ulam distance.

**Lemma 7.2.** *Let $r, n \in \mathbb{Z}_{>0}$, and $r|n$. Also let $\sigma, \pi \in \mathbb{S}_n$. Then*

$$\mathrm{d}_\circ^r(\sigma,\pi) = \min\{k \in \mathbb{Z}_{\geq 0} \mid \exists\ (\phi_1, \phi_2, \ldots, \phi_k) \text{ s.t. } \mathbf{m}_\sigma^r \cdot \phi_1\phi_2\cdots\phi_k = \mathbf{m}_\pi^r\}.$$

*Proof.* There exists a translocation $\phi \in \mathbb{S}_n$ such that $\ell(\mathbf{m}_\sigma^r \cdot \phi, \mathbf{m}_\pi^r) = \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) + 1$, since it is always possible to arrange one element with a single translocation. This then implies that $\min\{k \in \mathbb{Z} \mid \exists(\phi_1, \ldots, \phi_k) \text{ s.t. } \mathbf{m}_\sigma^r \cdot \phi_1 \cdots \phi_k = \mathbf{m}_\pi^r\} \leq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = \mathrm{d}_\circ^r(\sigma,\pi)$. At the same time, given $\ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) \leq n$, then for all translocations $\phi \in \mathbb{S}_n$, we have that $\ell(\mathbf{m}_\sigma^r \cdot \phi, \mathbf{m}_\pi^r) \leq \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) + 1$, since a single translocation can only arrange one element at a time. Therefore $\min\{k \in \mathbb{Z} \mid \exists(\phi_1, \ldots, \phi_k)$ s.t $\mathbf{m}_\sigma^r \cdot \phi_1 \cdots \phi_k = \mathbf{m}_\pi^r\} \geq n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) = \mathrm{d}_\circ^r(\sigma,\pi)$, by Lemma 7.1. □

Lemmas 7.1 and 7.2 answer the third question of relating the Ulam metric for permutations to the Ulam metric for multipermutations by allowing us to view the Ulam metric for $r$-regular multipermutations similarly to the way we view the Ulam metric for permutations; in terms of longest common subsequences or in terms of the minimum number of translocations. Another known property of the Ulam metric for permutations is left invariance, i.e. given $\sigma, \pi, \tau \in \mathbb{S}_n$, we have $\mathrm{d}_\circ(\sigma,\pi) = \mathrm{d}_\circ(\tau\sigma, \tau\pi)$. However, left invariance does not hold in general for multipermutations, as stated in the next lemma.

**Lemma 7.3.** *Let $r, n \in \mathbb{Z}_{>0}$, $r|n$, $n/r \geq 2$ and $r \geq 2$. Then there exist $\pi, \tau \in \mathbb{S}_n$ such that*

$$\mathrm{d}_\circ^r(e,\pi) \neq \mathrm{d}_\circ^r(\tau e, \tau\pi).$$

*Proof.* Let $n, r, \in \mathbb{Z}_{>0}$, $r|n$, $n/r \geq 2$, and $r \geq 2$. Define $\pi, \tau \in \mathbb{S}_n$ by

$$\pi := [\underbrace{2r, 2r-1, \ldots, 1}_{2r}, \underbrace{2r+1, 2r+2, \ldots, n}_{n-2r}] \text{ and}$$

$$\tau := [\underbrace{1, r+1, 2, r+2, \ldots, r, 2r}_{2r}, \underbrace{2r+1, 2r+2, \ldots, n}_{n-2r}].$$

First, consider $\mathrm{d}_\circ^r(e,\pi)$. Note that for $\mathbf{m}_e^r$ and $\mathbf{m}_\pi^r$, for any integer $i$ such that $2r < i \leq n$ we have $e(i) = \pi(i)$, which implies $\mathbf{m}_e^r(i) = \mathbf{m}_\pi^r(i)$. Meanwhile, the first $2r$ elements of $\mathbf{m}_e^r$ and $\mathbf{m}_\pi^r$ are $(\underbrace{1,1,\ldots,1}_{r}, \underbrace{2,2,\ldots,2}_{r})$ and $(\underbrace{2,2,\ldots,2}_{r}, \underbrace{1,1,\ldots,1}_{r})$ respectively, so that the longest common subsequence of the first $2r$ elements of $\mathbf{m}_e^r$ and $\mathbf{m}_\pi^r$ is comprised of $r$ 1's or $r$ 2's. Hence $\ell(\mathbf{m}_e^r, \mathbf{m}_\pi^r) = (n-2r)+r = n-r$, which by lemma 7.1 implies that $\mathrm{d}_\circ^r(e,\pi) = r \geq 2$.

Next, consider $\mathrm{d}_\circ^r(\tau e, \tau\pi)$. We have

$$\tau\pi = [\underbrace{2r, r, 2r-1, r-1, \ldots, r+1, 1}_{2r}, \underbrace{2r+1, 2r+2, \ldots, n}_{n-2r}].$$

For all integers $i$ such that $2r < i \leq n$, we have $\tau e(i) = \tau(i) = \tau\pi(i) \implies \mathbf{m}_{\tau e}^r(i) = \mathbf{m}_{\tau\pi}^r(i)$. Meanwhile, the first $2r$ elements of $\mathbf{m}_{\tau e}^r$ and $\mathbf{m}_{\tau\pi}$ are $(1,2,1,2,\ldots,1,2)$ and $(2,1,2,1,\ldots,2,1)$ respectively. Thus the longest common subsequence of the first $2r$ elements of $\mathbf{m}_{\tau e}^r$ and $\mathbf{m}_{\tau\pi}^r$ is any length $2r-1$ sequence of alternating 1's and 2's. Hence $\ell(\mathbf{m}_{\tau e}^r, \mathbf{m}_{\tau\pi}^r) = (n-2r)+(2r-1) = n-1$, which by lemma 7.1 implies that $\mathrm{d}_\circ^r(\tau e, \tau\pi) = 1$. $\square$

The fact that left invariance does not hold for the $r$-regular Ulam metric has implications on $r$-regular Ulam sphere sizes, defined and discussed in the next section. When left invariance does hold, as in the permutation case, sphere sizes do not depend upon the choice of center. On the other hand, when left invariance does not hold, as in the multipermutation case, then it is possible that sphere sizes differ depending upon the choice of center.

## 8 $r$-Regular Ulam Spheres

In this section we begin to analyze $r$-regular Ulam sphere sizes. The $r$-regular Ulam sphere sizes play an important role in understanding the potential code size for $\mathrm{MPC}_\circ(n,r,d)$ codes (recall that an $\mathrm{MPC}_\circ(n,r,d)$ code is a $\mathrm{MPC}(n,d)$ code with minimum $r$-regular Ulam distance $d$). For example, the well-known sphere-packing bounds and Gilbert-Varshamov type bounds rely on calculating, or at least bounding sphere sizes. In the case of permutations, recall that the the Ulam sphere $S(\sigma,t)$ centered at $\sigma$ of radius $t$ was defined as $S(\sigma,t) := \{\pi \in \mathbb{S}_n \mid \mathrm{d}_\circ(\sigma,\pi) \leq t\}$, which is equivalent by definition to the set $\{\pi \in \mathbb{S}_n \mid n - \ell(\sigma,\pi) \leq t\}$. In the case of $r$-regular multipermutations, for $r, n, t \in \mathbb{Z}_{>0}$ with $r|n$ and a given $\sigma \in \mathbb{S}_n$, we introduce the following analogous definition of a sphere.

**Definition.** Let $r, n \in \mathbb{Z}_{>0}$ and $r|n$. Also let $\sigma, \pi \in \mathbb{S}_n$. Define

$$S(\mathbf{m}_\sigma^r, t) := \{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) \mid \mathrm{d}_\circ^r(\sigma,\pi) \leq t\}$$

We call $S(\mathbf{m}_\sigma^r, t)$ the ***r*-regular Ulam sphere** centered at $\mathbf{m}_\sigma^r$ of radius $t$.

By Lemma 7.1, $S(\mathbf{m}_\pi^r, t) = \{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) \mid n - \ell(\mathbf{m}_\sigma^r, \mathbf{m}_\pi^r) \leq t\}$. It should be noted, however, that the notation $\mathbf{m}_\pi^r$ is a bit misleading because given $\mathbf{m}_\pi^r \in \mathcal{M}(\mathbb{S}_n)$, we cannot uniquely determine $\pi$. The $r$-regular Ulam sphere definition can also be viewed in terms of translocations. Lemma 7.2 implies that $S(\mathbf{m}_\sigma^r, t)$ is equivalent to $\{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) \mid \exists k \in [t]$ and $(\phi_1, \ldots, \phi_k)$ s.t. $\mathbf{m}_\sigma^r \cdot \phi_1 \cdots \phi_k = \mathbf{m}_\pi^r\}$.

Recall that the number of standard tableaux on a particular $\lambda \vdash n$ is denoted by $f^\lambda$. We denote by $K_r^\lambda$ (our own notation) the number of (not necessarily standard) Young tableaux on $\lambda \vdash n$ such that each $i \in [n/r]$ appears exactly $r$ times. The following remark is a straight-forward application of the RSK-correspondence, and states the relationship between $|S(\mathbf{m}_e^r, t)|$, $f^\lambda$, and $K_r^\lambda$.

**Remark 8.1.** Let $r, n \in \mathbb{Z}_{>0}$, $r|n$, $t \in \{0, \ldots, n-1\}$, and $\Lambda := \{\lambda \vdash n \mid \lambda_1 \geq n - t\}$. Then

$$|S(\mathbf{m}_e^r, t)| = \sum_{\lambda \in \Lambda} (f^\lambda)(K_r^\lambda).$$

Thus we have an explicit way of calculating sphere sizes centered at $\mathbf{m}_e^r$. In the permutation case, it was enough to consider sphere sizes centered at $e$ because sphere sizes did not depend on the choice of center. Unfortunately, in the case of multipermutations the choice of center has an impact on the size of the sphere, as is easily confirmed by an explicit calculation using Remark 8.1 and comparing the result to Proposition 8.9 at the end of this section. Therefore the applicability of the above formula is limited.

We begin to address the issue of sphere sizes depending on choice of center by considering the radius $t = 1$ case. To aid with calculating such sphere sizes, we first find it convenient to introduce (as our own definition) the following subset of the set of translocations.

**Definition.** Let $n \in \mathbb{Z}_{>0}$. Define $T_n := \{\phi(i,j) \in \mathbb{S}_n \mid i - j \neq 1\}$.
We call $T_n$ the **unique set of translocations**.

In other words, $T_n$ is the set of all translocations, except translocations of the form $\phi(i, i-1)$. We exclude translocations of this form because they can be modeled by translocations of the form $\phi(i-1, i)$, and are therefore redundant. We claim that the set $T_n$ is precisely the set of translocations needed to obtain all unique permutations within the Ulam sphere of radius 1 via multiplication. Moreover, there is no redundancy in the set, that is, there is no smaller set of translocations yielding the entire Ulam sphere of radius 1 when multiplied with a given center permutation. These facts are stated in the next two lemmas.

**Lemma 8.2.** *Given $n \in \mathbb{Z}_{>0}$ and $\sigma, \in \mathbb{S}_n$, then $S(\sigma, 1) = \{\sigma\phi \in \mathbb{S}_n \mid \phi \in T_n\}$.*

*Proof.* First note that $S(\sigma, 1) = \{\pi \in \mathbb{S}_n \mid \mathrm{d}_\circ(\sigma, \pi) \leq 1\} = \{\sigma\phi(i,j) \in \mathbb{S}_n \mid i, j \in [n]\}$. It is trivial that $T_n = \{\phi(i,j) \in \mathbb{S}_n \mid i - j \neq 1\} \subseteq \{\phi(i,j) \in \mathbb{S}_n \mid i, j \in [n]\}$, so $\{\sigma\phi \in \mathbb{S}_n \mid \phi \in T_n\} \subseteq S(\sigma, 1)$.

To see why $S(\sigma, 1) \subseteq \{\sigma\phi \in \mathbb{S}_n \mid \phi \in T_n\}$, consider any $\sigma\phi(i,j) \in \{\sigma\phi(i,j) \in \mathbb{S}_n \mid i, j \in [n]\} = S(\sigma, 1)$. If $i - j \neq 1$, then $\phi(i,j) \in T_n$, and thus $\sigma\phi(i,j) \in \{\sigma\phi \in \mathbb{S}_n \mid \phi \in T_n\}$. Otherwise, if $i - j = 1$, then $\sigma\phi(i,j) = \sigma\phi(j,i)$, and $i - j = 1 \implies j - i = -1 \neq 1$, so $\phi(j,i) \in T_n$. Hence $\sigma\phi(i,j) = \sigma\phi(j,i) \in \{\sigma\phi \in \mathbb{S}_n \mid \phi \in T_n\}$. □

**Lemma 8.3.** *Given $n \in \mathbb{Z}_{>0}$ and $\sigma, \in \mathbb{S}_n$, then $|T_n| = |S(\sigma, 1)|$*

*Proof.* By Lemma 3.4, $|S(\sigma, 1)| = 1 + (n-1)^2$. On the other hand, $|T_n| = |\{\phi(i,j) \in \mathbb{S}_n \mid i - j \neq 1\}|$. Now if $i = 1$, then there are $n$ values $j \in [n]$ such that $i - j \neq 1$. Otherwise, if $i \in [n]$ but $i \neq 1$, then there are $n - 1$ values $j \in [n]$ such that $i - j \neq 1$. However, for all $i, j \in [n]$, $\phi(i,i) = \phi(j,j) = e$ so that there are $n - 1$ redundancies. Therefore $|T_n| = n + (n-1)(n-1) - (n-1) = (n-1)^2 + 1 = 1 + (n-1)^2$. □

Although the Ulam sphere centered at $\sigma \in \mathbb{S}_n$ of radius 1 can be characterized by all permutations obtainable by applying (multiplying on the right) a translocation to $\sigma$, the previous two lemmas show that some translocations are redundant. That is, there are translocations $\phi_1 \neq \phi_2$ such that $\sigma\phi_1 = \sigma\phi_2$. In the case of permutations, the set $T_n$, however, has no such redundancies. If $\phi_1, \phi_2 \in T_n$, then $\sigma\phi_1 = \sigma\phi_2 \implies \phi_1 = \phi_2$. Alternatively, in the case of multipermutations, the set $T_n$ can generally be shrunken further to exclude redundancies.

Notice that for $r, n \in \mathbb{Z}_{>0}$, $r|n$, and $\sigma \in \mathbb{S}_n$ that the sphere $S(\mathbf{m}_\sigma^r, 1) = \{\mathbf{m}_\pi^r \in \mathcal{M}_r(\mathbb{S}_n) \mid \exists\phi$ s.t. $\mathbf{m}_\sigma^r \cdot \phi = \mathbf{m}_\pi\} = \{\mathbf{m}_\sigma^r \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) \mid \phi \in T_n\}$. However, it is possible that there exist $\phi_1, \phi_2 \in T_n$ such that $\phi_1 \neq \phi_2$, but $\mathbf{m}_\sigma^r \cdot \phi_1 = \mathbf{m}_\sigma^r \cdot \phi_2$. In such an instance we may refer to either $\phi_1$ or $\phi_2$ as a **duplicate**

**translocation** for $\mathbf{m}^r_\sigma$. If we remove all duplicate translocations for $\mathbf{m}^r_\sigma$ from $T_n$, then the resulting set will have the same cardinality as the $r$-regular Ulam sphere of radius 1 centered at $\mathbf{m}^r_\sigma$. The next definition (our own definition) is a standard set of duplicate translocations.

**Definition** ($D(\mathbf{m})$, set of standard duplicate translocations). Given $n \in \mathbb{Z}_{>0}$ and a tuple $\mathbf{m} \in \mathbb{Z}^n$, define

$$D(\mathbf{m}) := \{\phi(i,j) \in T_n \backslash \{e\} \mid \mathbf{m}(i) = \mathbf{m}(j) \vee \mathbf{m}(i) = \mathbf{m}(i-1)\}$$

We call $D(\mathbf{m})$ the **set of standard duplicate translocations** for $\mathbf{m}$.

If we take an $r$-regular multipermutation $\mathbf{m}^r_\sigma$, then removing the general set of duplications from $T_n$ equates to removing a set of duplicate translocations. These duplications come in two varieties.

The first variety corresponds to the first condition of the $D(\mathbf{m})$ definition, when $\mathbf{m}(i) = \mathbf{m}(j)$. For example, if $\sigma \in \mathbb{S}_6$ such that $\mathbf{m}^2_\sigma = [1,3,2,2,3,1]$, then we have $\mathbf{m}^2_\sigma\phi(1,5) = [3,2,2,3,1,1] = \mathbf{m}^2_\sigma\phi(1,6)$, since $\mathbf{m}^2_\sigma(2) = 3 = \mathbf{m}^2_\sigma(4)$. This is because moving the first 1 to the left or to the right of the last 1 results in the same tuple.

The second variety corresponds to the second condition of the of $D(\mathbf{m})$ definition above, when $\mathbf{m}(i) = \mathbf{m}(i-1)$. For example, if $\mathbf{m}^2_\sigma = [1,3,2,2,3,1]$ as before, then for all $j \in [6]$, we have $\mathbf{m}^2_\sigma \cdot \phi(3,j) = \mathbf{m}^2_\sigma \cdot \phi(4,j)$. This is because any translocation that deletes and inserts the second of the two adjacent 2's does not result in a different tuple when compared to deleting and inserting the first of the two adjacent 2's.

**Lemma 8.4.** *Let $r, n \in \mathbb{Z}_{>0}$, $r|n$, and $\sigma \in \mathbb{S}_n$. Then $S(\mathbf{m}^r_\sigma, 1) = \{\mathbf{m}^r_\sigma \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) \mid \phi \in T_n \backslash D(\mathbf{m}^r_\sigma)\}$.*

*Proof.* Let $r, n \in \mathbb{Z}_{>0}$, $r|n$, and $\sigma \in \mathbb{S}_n$. First note that $S(\mathbf{m}^r_\sigma, 1) = \{\mathbf{m}^r_\sigma\phi \in \mathcal{M}_r(\mathbb{S}_n) \mid \phi \in T_n\}$. Hence it suffices to show that for all $\phi(i,j) \in D(\mathbf{m}^r_\sigma)$, there exists some

$i', j' \in [n]$ such that $\phi(i',j') \in T_n \backslash D(\mathbf{m}^r_\sigma)$ and $\mathbf{m}^r_\sigma\phi(i,j) = \mathbf{m}^r_\sigma\phi(i',j')$. We proceed by dividing the proof into two main cases. Case I is when $(\mathbf{m}^r_\sigma(i) \neq \mathbf{m}^r_\sigma(i-1)$ or $i = 1$. Case II is when $(\mathbf{m}^r_\sigma(i) = \mathbf{m}^r_\sigma(i-1)$.

Case I (when $(\mathbf{m}^r_\sigma(i) \neq \mathbf{m}^r_\sigma(i-1)$ or $i = 1$) can be split into two subcases:

Case IA: $i < j$

Case IB: $i > j$.

We can ignore the instance when $i = j$, since $\phi(i,j) \in D(\mathbf{m}^r_\sigma)$ implies $i \neq j$. For case IA, if for all $p \in [i,j]$ (for $a, b \in \mathbb{Z}$ with $a < b$, the notation $[a,b] := \{a, a+1, \ldots, b\}$) we have $\mathbf{m}^r_\sigma(i) = \mathbf{m}^r_\sigma(p)$, then $\mathbf{m}^r_\sigma\phi(i,j) = \mathbf{m}^r_\sigma e$. Thus setting $i' = j' = 1$ yields the desired result. Otherwise, if there exists $p \in [i,j]$ such that $\mathbf{m}^r_\sigma(i) \neq \mathbf{m}^r_\sigma(p)$, then let $j^* := j - \min\{k \in \mathbb{Z}_{>0} \mid \mathbf{m}^r_\sigma(i) \neq \mathbf{m}^r_\sigma(j-k)\}$. Then $\phi(i,j^*) \in T_n \backslash D(\mathbf{m}^r_\sigma)$ and $\mathbf{m}^r_\sigma\phi(i,j) = \mathbf{m}^r_\sigma\phi(i,j^*)$. Thus setting $i' = i$ and $j' = j^*$ yields the desired result. Case IB is similar to Case IA.

Case II (when $\mathbf{m}^r_\sigma(i) = \mathbf{m}^r_\sigma(i-1)$), can also be divided into two subcases.

Case IIA: $i < j$

Case IIB: $i > j$.

As in Case I, we can ignore the instance when $i = j$. For Case IIA, if for all $p \in [i,j]$ we have $\mathbf{m}^r_\sigma(i) = \mathbf{m}^r_\sigma(p)$, then $\mathbf{m}^r_\sigma(\phi(i,j) = \mathbf{m}^r_\sigma(e)$, so setting $i = j = 1$ achieves the desired result. Otherwise, if there exists $p \in [i,j]$ such that $\mathbf{m}^r_\sigma(i) \neq \mathbf{m}^r_\sigma(p)$, then let $i^* := i - \min\{k \in \mathbb{Z}_{>0} \mid (\mathbf{m}^r_\sigma(i) \neq \mathbf{m}^r_\sigma(i-k-1)) \vee (i-k=1)\}$. Then $\mathbf{m}^r_\sigma\phi(i,j) = \mathbf{m}^r_\sigma\phi(i^*,j)$ and either one of the following is true: 1) $\phi(i^*,j) \notin D_{i^*}(\mathbf{m}^r_\sigma) \implies \phi(i^*,j) \notin D(\mathbf{m}^r_\sigma)$, so set $i' = i^*$ and $j' = j$, or 2) by Case IA there exist $i', j' \in [n]$ such that $\phi(i',j') \in T_n \backslash D(\mathbf{m}^r_\sigma)$ and $\mathbf{m}^r_\sigma\phi(i',j') = \mathbf{m}^r_\sigma\phi(i^*,j) = \mathbf{m}^r_\sigma\phi(i,j)$. Case IIB is similar to Case IIA. □

While Lemma 8.4 shows that $D(\mathbf{m}^r_\sigma)$ is a set of duplicate translocations for $\mathbf{m}^r_\sigma$, we have not shown that $T_n \backslash D(\mathbf{m}^r_\sigma)$ is the set of minimal size having the quality that $S(\mathbf{m}^r_\sigma, 1) = \{\mathbf{m}^r_\sigma \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) \mid \phi \in T_n \backslash D(\mathbf{m}^r_\sigma)\}$. In fact it is not minimal. In some instances it is possible to remove further duplicate translocations to reduce the set size. For a sequence $\mathbf{m} \in \mathbb{Z}^n$, we define (also our own definition) the following additional set of duplications.

**Definition** ($E(\mathbf{m})$, set of alternating duplicate translocations). Given $n \in \mathbb{Z}_{>0}$ and a tuple $\mathbf{m} \in \mathbb{Z}^n$, define

$$E(\mathbf{m}) := \{\phi(i,j) \in T_n \backslash D(\mathbf{m}) \mid i < j \text{ and } \exists\, k \in [i, j-1] \text{ s.t. } (\phi(j,k) \in T_n \backslash D(\mathbf{m})) \wedge (\mathbf{m} \cdot \phi(i,j) = \mathbf{m} \cdot \phi(j,k))\}.$$

We call $E(\mathbf{m})$ the **set of alternating duplicate translocations** for $\mathbf{m}$ because it is only nonempty when $\mathbf{m}$ contains an alternating substring of length greater than 4. For $k \in \mathbb{Z}_{\geq 0}$, an alternating substring of $\mathbf{m}$ is a tuple of the form $(\mathbf{m}(i), \mathbf{m}(i+1), \ldots \mathbf{m}(i+k)) \in \mathbb{Z}^{k+1}$ such that for all even values of $l \in [k]$, $\mathbf{m}(i) = \mathbf{m}(i+l)$ while for all odd values of $l' \in [k]$, $(\mathbf{m}(i+1) = \mathbf{m}(i+l')) \wedge (\mathbf{m}(i) \neq \mathbf{m}(i+l'))$.

Finally, we define the general set of duplications. The lemma that follows the definition also shows that for $\sigma \in \mathbb{S}_n$, removing the set $D^*(\mathbf{m}^r_\sigma)$ from $T_n$ removes all duplicate translocations associated with $\mathbf{m}^r_\sigma$.

**Definition** ($D^*(\mathbf{m})$, duplication set). Given $n \in \mathbb{Z}_{>0}$ and a tuple $\mathbf{m} \in \mathbb{Z}^n$, define

$$D^*(\mathbf{m}) := D(\mathbf{m}) \cup E(\mathbf{m}).$$

We call $D^*(\mathbf{m})$ the **duplication set** for $\mathbf{m}$.

**Lemma 8.5.** *Let $r, n \in \mathbb{Z}$, $r|n$, $\sigma \in \mathbb{S}_n$, and $\phi_1, \phi_2 \in T_n \backslash D^*(\mathbf{m}^r_\sigma)$. Then $\phi_1 = \phi_2$ if and only if $\mathbf{m}^r_\sigma \cdot \phi_1 = \mathbf{m}^r_\sigma \cdot \phi_2$.*

*Proof.* Let $r, n \in \mathbb{Z}$, $r|n$, $\sigma \in \mathbb{S}_n$, and $\phi_1, \phi_2 \in T_n \backslash D^*(\mathbf{m}^r_\sigma)$. If $\phi_1 = \phi_2$ then $\mathbf{m}^r_\sigma \phi_1 = \mathbf{m}^r_\sigma \phi_2$ trivially. It remains to prove that $\mathbf{m}^r_\sigma \phi_1 = \mathbf{m}^r_\sigma \phi_2 \implies \phi_1 = \phi_2$. We proceed by contrapositive. Suppose that $\phi_1 \neq \phi_2$. We want to show that $\mathbf{m}^r_\sigma \phi_1 \neq \mathbf{m}^r_\sigma \phi_2$. Let $\phi_1 := \phi(i_1, j_1)$ and $\phi_2 := \phi(i_2, j_2)$. The remainder of the proof can be split into two main cases: Case I is if $i_1 = i_2$ and Case II is if $i_1 \neq i_2$.

Case I (when $i_1 = i_2$), can be further divided into two subcases:

$$\begin{aligned} &\text{Case IA; } \mathbf{m}^r_\sigma(i_1) = \mathbf{m}^r_\sigma(i_1 - 1) \\ &\text{Case IB: } \mathbf{m}^r_\sigma(i_1) \neq \mathbf{m}^r_\sigma(i_1 - 1). \end{aligned}$$

Case IA is easy to prove. We have $D^*_{i_1}(\mathbf{m}^r_\sigma) = D^*_{i_2}(\mathbf{m}^r_\sigma) = \{\phi(i_1, j) \in T_n \backslash \{e\} \mid j \in [n]\}$, so $\phi_1 = e = \phi_2$, a contradiction. For Case IB, we can first assume without loss of generality that $j_1 < j_2$ and then split into the following smaller subcases:

$$\begin{aligned} \text{i) } &(j_1 < i_1) \wedge (j_2 > i_1) \\ \text{ii) } &(j_1 < i_1) \wedge (j_2 \leq i_1) \\ \text{iii) } &(j_1 > i_1) \wedge (j_2 > i_1) \\ \text{iv) } &(j_1 > i_1) \wedge (j_2 \leq i_1). \end{aligned}$$

However, subcase iv) is unnecessary since it was assumed that $j_1 < j_2$, so $j_1 > i_1 \implies j_2 > j_1 > i_1$. Subcase ii) can also be reduced to $(j_1 < i_1) \wedge (j_2 < i_1)$ since $j_2 \neq i_2 = i_1$. Each of the remaining subcases is proven by noting that there is some element in the multipermutation $\mathbf{m}^r_\sigma \phi_1$ that is necessarily different from $\mathbf{m}^r_\sigma \phi_2$. For example, in subcase i), we have $\mathbf{m}^r_\sigma \phi_1(j_1) = \mathbf{m}^r_\sigma(i_1) \neq \mathbf{m}^r_\sigma(j_1) = \mathbf{m}^r_\sigma \phi_2(j_1)$. Subcases ii) and iii) are solved similarly.

Case II (when $i_1 \neq i_2$) can be divided into three subcases:

$$\begin{aligned} &\text{Case IIA: } (\mathbf{m}^r_\sigma(i_1) = \mathbf{m}^r_\sigma(i_1 - 1) \wedge \mathbf{m}^r_\sigma(i_2) = \mathbf{m}^r_\sigma(i_2 - 1)), \\ &\text{Case IIB: either } (\mathbf{m}^r_\sigma(i_1) = \mathbf{m}^r_\sigma(i_1 - 1) \wedge \mathbf{m}^r_\sigma(i_2) \neq \mathbf{m}^r_\sigma(i_2 - 1)) \\ &\qquad\qquad \text{or } (\mathbf{m}^r_\sigma(i_1) \neq \mathbf{m}^r_\sigma(i_1 - 1) \wedge \mathbf{m}^r_\sigma(i_2) = \mathbf{m}^r_\sigma(i_2 - 1)), \\ &\text{Case IIC: } (\mathbf{m}^r_\sigma(i_1) \neq \mathbf{m}^r_\sigma(i_1 - 1) \wedge \mathbf{m}^r_\sigma(i_2) \neq \mathbf{m}^r_\sigma(i_2 - 1)). \end{aligned}$$

Case IIA is easily solved by mimicking the proof of Case IA. Case IIB is also easily solved as follows. First, without loss of generality, we assume that $\mathbf{m}^r_\sigma(i_1) = \mathbf{m}^r_\sigma(i_1 - 1) \wedge \mathbf{m}^r_\sigma(i_2) \neq \mathbf{m}^r_\sigma(i_2 - 1)$. Then $D^*_{i_1}(\mathbf{m}^r_\sigma) = \{\phi(i_1, j) \in T_n \backslash \{e\} \mid j \in [n]\}$, so $\phi_1 = e$. Therefore we have $\mathbf{m}^r_\sigma \phi_1(j_2) = \mathbf{m}^r_\sigma(j_2) \neq \mathbf{m}^r_\sigma(i_2) = \mathbf{m}^r_\sigma \phi_2(i_2 - 1)$.

Finally, for Case IIC, without loss of generality we may assume that $i_1 < i_2$ and then split into the following four subcases:

$$\begin{aligned} &\text{i) } (j_1 < i_2) \wedge (j_2 \geq i_2) \\ &\text{ii) } (j_1 < i_2) \wedge (j_2 < i_2) \\ &\text{iii) } (j_1 \geq i_2) \wedge (j_2 \geq i_2) \\ &\text{iv) } (j_1 \geq i_2) \wedge (j_2 < i_2). \end{aligned}$$

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However, since $\phi(i_2, j_2) \in T_n \backslash D^*(\mathbf{m}^r_\sigma)$ implies $i_2 \neq j_2$, subcases i) and iii) can be reduced to $(j_1 < i_2) \wedge (j_2 > i_2)$ and $(j_1 \geq i_2) \wedge (j_2 > i_2)$ respectively. For subcase i), we have $\mathbf{m}^r_\sigma \phi_1(j_1) = \mathbf{m}^r_\sigma(i_1) \neq \mathbf{m}^r_\sigma(j_1) = \mathbf{m}^r_\sigma \phi_2(j_1)$. Subcases ii) and iii) are solved in a similar manner. For subcase iv), if $j_1 > i_2$, then $\mathbf{m}\phi_1(j_1) = \mathbf{m}(i_1) \neq \mathbf{m}^r_\sigma(j_1) = \mathbf{m}^r_\sigma \phi_2(j_1)$. Otherwise, if $j_1 = i_2$, then $\phi_1 = \phi(i_1, i_2)$ and $\phi_1 = \phi(i_2, j_2)$. Thus if $\mathbf{m}^r_\sigma \phi_1 = \mathbf{m}^r_\sigma \phi_2$ then $\phi_1 \in D^*_{i_1}(\mathbf{m}^r_\sigma)$, which implies that $\phi_1 \notin T_n \backslash D^*(\mathbf{m}^r_\sigma)$, a contradiction. □

Lemma 8.5 implies that we can calculate $r$-regular Ulam sphere sizes of radius 1 whenever we can calculate the appropriate duplication set. This calculation can be simplified by noting that for a sequence $\mathbf{m} \in \mathbb{Z}^n$ that $D(\mathbf{m}) \cap E(\mathbf{m}) = \varnothing$ (by the definition of $E(\mathbf{m})$) and then decomposing the duplication set into these components. The next proposition and corollary are another main result of this paper.

**Proposition 8.6.** *Let $r, n \in \mathbb{Z}$, $r|n$, $\sigma \in \mathbb{S}_n$. Then $|S(\mathbf{m}^r_\sigma, 1)| = (n-1)^2 + 1 - |D(\mathbf{m}^r_\sigma)| - |E(\mathbf{m}^r_\sigma)|$.*

*Proof.* By the definition of $D^*(\mathbf{m}^r_\sigma)$ and lemma 8.4, $\{\mathbf{m}^r_\sigma \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) \mid \phi \in T_n \backslash D^*(\mathbf{m}^r_\sigma)\} = \{\mathbf{m}^r_\sigma \cdot \phi \in \mathcal{M}_r(\mathbb{S}_n) \mid \phi \in T_n \backslash D(\mathbf{m}^r_\sigma)\} = S(\mathbf{m}^r_\sigma, 1)$. This implies $|T_n \backslash D^*(\mathbf{m}^r_\sigma)| \geq |S(\mathbf{m}^r_\sigma, 1)|$. By lemma 8.5, for $\phi_1, \phi_2 \in T_n \backslash D^*(\mathbf{m}^r_\sigma)$, if $\phi_1 \neq \phi_2$, then $\mathbf{m}^r_\sigma \cdot \phi_1 \neq \mathbf{m}^r_\sigma \cdot \phi_2$. Hence we have $|T_n \backslash D^*(\mathbf{m}^r_\sigma)| \leq |S(\mathbf{m}^r_\sigma, 1)|$, which implies that $|T_n \backslash D^*(\mathbf{m}^r_\sigma)| = |S(\mathbf{m}^r_\sigma, 1)|$. It remains to show that $|T_n \backslash D^*(\mathbf{m}^r_\sigma)| = (n-1)^2 + 1 - |D(\mathbf{m}^r_\sigma)| - |E(\mathbf{m}^r_\sigma)|$. This is an immediate consequence of the fact that $|T_n| = (n-1)^2 + 1$ and $D(\mathbf{m}^r_\sigma) \cap E(\mathbf{m}^r_\sigma) = \varnothing$. □

Proposition 8.6 gives an explicit way to calculate radius $t = 1$ sphere sizes for any choice of multipermutation (This is not limited to $r$-regular multipermutations). The proposition also can be used to obtain an upper bound on the size of $r$-regular Ulam spheres. We will then show that this upper bound is tight (except for one special case) by demonstrating an $r$-regular Ulam sphere that achieves this bound.

**Corollary 8.7.** *Let $r, n \in \mathbb{Z}$, $r|n$, and $\sigma \in \mathbb{S}_n$. Then $|S(\mathbf{m}^r_\sigma, 1)| \leq ((n-1)^2 + 1) - (r-1)n$.*

*Proof.* First, note that if $n/r = 1$, then $|S(\mathbf{m}^r_\sigma, 1)| = 1$ and the corollary holds trivially. Assume then that $n/r \geq 2$. By Proposition 8.6, $|S(\mathbf{m}^r_\sigma, 1)| = ((n-1)^2+1) - |D(\mathbf{m}^r_\sigma)| - |E(\mathbf{m}^r_\sigma)| \leq ((n-1)^2+1) - |D(\mathbf{m}^r_\sigma)|$.

Notice that if $\mathbf{m}^r_\sigma(i) \neq \mathbf{m}^r_\sigma(i-1)$ or if $i = 1$ then $|D(\mathbf{m}^r_\sigma)| = r - 1$. This is because in such instance the first condition of the definition of $D(\mathbf{m})$ applies, and each element of an $r$-regular multipermutation has $r - 1$ other elements equal to itself.

Otherwise $|D(\mathbf{m}^r_\sigma)| = n-2$ since we are in the second condition of $D(\mathbf{m}^r_\sigma)$, where $D(\mathbf{m}^r_\sigma)$ is comprised of any $\phi(i,j)$ as long as $i \neq j$ and $i - j \neq 1$. Since $n/r \geq 2$, it follows that $n \geq 2r$, so $n-2 \geq 2r-2 = 2(r-1)$. Therefore $|D(\mathbf{m}^r_\sigma| \geq (r-1)n$, which implies $|S(\mathbf{m}^r_\sigma, 1)| \leq ((n-1)^2 + 1) - (r-1)n$. □

Extending the concept of perfect permutation codes discussed in earlier sections, we define a perfect multipermutation code.

**Definition.** Let $C$ be an MPC$(n,r)$ code. Then $C$ is a perfect $t$-error correcting code if and only if for all $\pi \in S_n$, there exists a unique $\mathbf{m}^r_\sigma \in \mathcal{M}_r(C)$ such that $\mathbf{m}^r_\pi \in |S(\mathbf{m}^r_\sigma, t)|$.

In such instances we call $C$ a **perfect $t$-error correcting** MPC$(n,r)$ **code**.

**Remark 8.8.** With this definition the upper bound of Corollary 8.7 implies a lower bound on a perfect single-error correcting MPC$(n,r)$ code, namely if $C$ is a perfect single-error correcting MPC$(n,r)$ code, then

$$|C|_r \geq \frac{n!}{(r!)^{n/r} \cdot (((n-1)^2 + 1) - (r-1)n)}.$$

Furthermore, the upper bound of Corollary 8.7 is achievable with the proper choice of center. Let $\omega^* \in \mathbb{S}_n$ be defined as follows: $\omega^*(i) := ((i-1) \mod (n/r))r + \lceil ir/n \rceil$ and $\omega^* := [\omega^*(1), \omega^*(2), \ldots \omega^*(n)]$. With this definition, for all $i \in [n]$, we have $\mathbf{m}^r_{\omega^*}(i) = i \mod (n/r)$ For example, if $r = 3$ and $n = 12$, then $\omega^* = [1, 4, 7, 10, 2, 5, 8, 11, 3, 6, 9, 12]$ and $\mathbf{m}^r_{\omega^*} = [1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4]$. Except for one special case (when $r = 2$), the ball centered at $\mathbf{m}^r_{\omega^*}$ achieves the bound of Corollary 8.7.

**Proposition 8.9.** *Let $n/r \neq 2$. Then $|S(\mathbf{m}^r_{\omega^*}, 1)| = ((n-1)^2 + 1) - (r-1)n$.*

*Proof.* Assume $n/r \neq 2$. First note as in the proof of Corollary 8.7 that if $n/r = 1$ then the proposition holds trivially.

Next, assume that $n/r \geq 3$. Then $E(\mathbf{m}^r_{\omega^*}) = \varnothing$. To see why this is true, first note that if $\phi(i,j) \in E(\mathbf{m}^r_{\omega^*})$, then $i < j$ and there exists some $k \in [i, j-1]$ such that $\mathbf{m}^r_{\omega^*} \cdot \phi(i,j) = \mathbf{m}^r_{\omega^*} \cdot \phi(j,k)$. In such an instance, if $k \neq i$, then $\mathbf{m}^r_{\omega^*} \cdot \phi(i,j)(i) = \mathbf{m}^r_{\omega^*}(i+1)$ since applying $\phi(i,j)$ deletes $\mathbf{m}^r_{\omega^*}(i)$ from the $i$th position and inserts it in the $j$th position, meanwhile shifting $\mathbf{m}^r_{\omega^*}(i+1)$ to the $i$th position.

On the other hand, $\mathbf{m}^r_{\omega^*} \cdot \phi(j,k)(i) = \mathbf{m}^r_{\omega^*}(i)$ since both $k$ and $j$ are greater than $i$ so that $i$ is unaffected by the translocation. Otherwise, if $k = i$, then $\mathbf{m}^r_{\omega^*} \cdot \phi(i,j) = \mathbf{m}^r_{\omega^*} \cdot \phi(j,k)$ implies $\mathbf{m}^r_{\omega^*}(i) = \mathbf{m}^r_{\omega^*}(i+2)$, but this is impossible whenever $n/r \geq 3$. Therefore by Proposition 8.6 we have $|S(\mathbf{m}^r_{\omega^*}, 1)| = (n-1)^2 + 1 - |D(\mathbf{m}^r_{\omega^*})|$. Since there does not exist $i \in [n]$ such that $\mathbf{m}^r_{\omega^*}(i) = \mathbf{m}^r_{\omega^*}(i-1)$, we have $|D(\mathbf{m}^r_{\omega^*})| = (r-1)n$. □

In this section we began considering the problem of calculating the size of $r$-regular Ulam spheres of radius 1. We showed (Proposition 8.6) that for a given $r$-regular multipermutation $\mathbf{m}^r_\sigma$, the problem essentially reduces to calculating $E(\mathbf{m}^r_\sigma)$ and $D(\mathbf{m}^r_\sigma)$. It remains also to consider arbitrary radii.

# 9 Conclusion

This paper first considered and answered two questions. The first question concerned Ulam sphere sizes and the second concerned the possibility of perfect codes. It was shown that Ulam sphere sizes can be calculated explicitly for reasonably small radii using an application of the RSK-correspondence. It was then shown, partially using the afforementioned sphere-calculation method, that nontrivial perfect Ulam permutation codes do not exist.

Following the discussion of permutation codes, the multipermutation code case was considered next, and two more questions were addressed. Answering the third question, similarities between the Ulam metric for permutations and the $r$-regular Ulam metric for multipermutations were shown, resulting in a simplification of the latter. Finally, the final question of calculating $r$-regular Ulam spheres was addressed for the cases when the center is $\mathbf{m}^r_e$ or when the radius $t = 1$.

Many remaining problems remain. One problem is to find a method for calculating $r$-regular Ulam spheres for more general parameters. Our current work began to show how to calculate sizes for any radius when the center is $\mathbf{m}^r_e$ (using Young tableaux) or for any center when the radius is 1, but not for general parameters. A general formula for any center or radii, or at least bounds on general sphere sizes, would help in understanding bounds on the size of multipermutation Ulam codes and the possibility of perfect multipermutation codes.

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